BH Astrophys Ch6.6

The Maxwell equations – how charges produce fields

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = 4\pi\rho$$

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

Total of 8 equations, but only 6 independent variables (3 components each for E,B)

Where are the two extra equations hidden?

$$-\frac{\partial(\nabla \cdot \boldsymbol{D})}{\partial t} = 4\pi(\nabla \cdot \boldsymbol{J})$$

$$-\frac{\partial \rho_q}{\partial t} = \nabla \cdot \boldsymbol{J}$$

Conservation of (electric) charge

See Jackson 3ed Ch 6.11 for more about this

$$\frac{\partial(\nabla \cdot \boldsymbol{B})}{\partial t} = 0$$

"Constrained transport" of magnetic field
Conservation of magnetic monopoles (equal zero in our universe)

Therefore the Maxwell equations actually have in-built conservation of electric and magnetic charges! (therefore total independent equations are 6 only)

Energy momentum equations – How fields affect particles

Work done by EM field on charges $$\left[\frac{d\mathcal{E}}{dt}\right]_{\mathrm{EM}} = q\,\boldsymbol{V} \cdot \boldsymbol{E}$$

Lorentz force equation $$\boldsymbol{F}_{\mathrm{EM}} = q\left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \times \boldsymbol{B}\right)$$

For fluids, they translate to

$$\rho\,\dot{q} = \boldsymbol{J} \cdot \boldsymbol{E} \qquad \mathfrak{F}_{\mathrm{EM}} = \rho_q \boldsymbol{E} + \frac{1}{c}\boldsymbol{J} \times \boldsymbol{B}$$

$\dot{q}$ is the local heating (or cooling) rate per unit mass

Recall energy equation of fluid $$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon\,\boldsymbol{V}) = -p\,\nabla \cdot \boldsymbol{V} + \rho\,\dot{\mathrm{q}}$$

The scalar and vector potentials

Given the Maxwell equations, $\nabla \cdot \boldsymbol{B} = 0$ and $\frac{\partial \boldsymbol{B}}{c\partial t} + \nabla \times \boldsymbol{E} = 0$

One can define a scalar potential ϕ and a vector potential $\vec{A}$ which have the relations with $\vec{E}$ and $\vec{B}$ as

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

However, we can see that any change of ϕ and $\vec{A}$ that follow

$$\phi' = \phi + \frac{1}{c}\frac{\partial Z}{\partial t} \qquad \vec{A}' = \vec{A} + \nabla Z$$

Will still give the same $\vec{E}$ and $\vec{B}$ fields

This is called "gauge invariance", we have the freedom to choose some Z that would make problems easier.

The Lorenz Gauge

Applying the definitions $\vec{B} = \nabla \times \vec{A}$ $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

One can transform the two source equations $\nabla \cdot \vec{D} = 4\pi\rho$ $\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

into

$$\nabla^2 \Phi + \frac{\partial(\nabla \cdot \boldsymbol{A})}{c\,\partial t} = -4\pi\rho_q$$

$$\nabla^2 \boldsymbol{A} - \frac{\partial^2 \boldsymbol{A}}{c^2\,\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{\partial \Phi}{c\,\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{J}$$

Often when the problem involves waves, a convenient gauge is the “Lotenz Gauge”

$\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$ Which, through $\phi' = \phi + \frac{1}{c}\frac{\partial Z}{\partial t}$, $\vec{A}' = \vec{A} + \nabla Z$ also tells us that $-\frac{1}{c^2}\frac{\partial^2 Z}{\partial t^2} + \nabla^2 Z = 0$

Applying the Lorenz Gauge condition, the two potential equations become

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 4\pi\rho \qquad \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}$$

This is not the Lorentz as in Lorentz transformations! Ludvig Lorenz (1829~1891) Hendrik Lorentz (1853~1928)

Linearly polarized plane EM waves in vacuum

In vacuum, the wave equations become $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0$ $\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}$

Which gives

$$\boldsymbol{\Phi} = \Re\{\varphi \exp[\mathrm{i}(kz - \omega t)]\}$$
$$\boldsymbol{A} = \Re\{\mathcal{A} \exp[\mathrm{i}(kz - \omega t)]\}$$
$$\boldsymbol{E} = \Re\{\mathcal{E} \exp[\mathrm{i}(kz - \omega t)]\}$$
$$\boldsymbol{B} = \Re\{\mathcal{B} \exp[\mathrm{i}(kz - \omega t)]\}$$
$$\omega^2 = k^2 c^2$$

The Lotenz gauge condition $\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$ gives $\mathcal{A}^z = \pm\varphi$

Since Z used in the gauge transforms also satisfies $-\frac{1}{c^2}\frac{\partial^2 Z}{\partial t^2} + \nabla^2 Z = 0$

It also has the wave solution $Z = \Re\{\mathcal{Z} \exp[\mathrm{i}(kz - \omega t)]\}$

$$\phi' = \phi + \frac{1}{c}\frac{\partial Z}{\partial t}$$
$$\vec{A}' = \vec{A} + \nabla Z$$

Then tells us that

$$\varphi' = \varphi \pm \mathrm{i}k\mathcal{Z}$$
$$\mathcal{A}'^x = \mathcal{A}^x$$
$$\mathcal{A}'^y = \mathcal{A}^y$$
$$\mathcal{A}'^z = \mathcal{A}^z \pm \mathrm{i}k\mathcal{Z}$$

Building the spacetime compatible form of electromagnetism

To work in spacetime, we need equations to be in tensor form such that they will be valid in any frame, in harmony with the principle of relativity.

It is therefore crucial to rewrite the whole set of equations in 4-form, either with 4-vectors or tensors.

Previously, we started off with the electric and magnetic fields from the Maxwell's equations and finally came to another fully equivalent form in describing things – the potentials.

Here, we will do the reverse, we work out the potential in 4-form then find some suitable tensor to fit in the electric and magnetic fields.

The current 4-vector

Recall the charge continuity equation $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Observing closely, we see that if we define $J^{\mu} = \left(\rho, \vec{J}\right)$, then using $x^{\mu} = (t, x, y, z)$, the continuity equation can very simply be written as

$$\frac{\partial J^{\mu}}{\partial x^{\mu}} \equiv J^{\mu}{}_{,\mu} = \partial_{\mu} J^{\mu} = 0$$

(contracting a vector and a one-form gives a scalar)

What this says is that J^{μ} is the correct 4-vector that compatible with space-time transforms!

Note:

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} = \left(-\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

The potential 4-vector

Recall the wave equations $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 4\pi\rho \qquad \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}$

We can see that in both cases we have the operator $\frac{\partial^2}{\partial t^2} - \nabla^2$ acting on either ϕ or $\vec{A}$.

Is that operator also related to some 4-vector?

Apparently, Yes! It is the square of the gradient operator!

$$\partial^\mu \partial_\mu = -(\frac{\partial^2}{\partial t^2} - \nabla^2)$$

Then, since $J^\mu = \left(\rho, \vec{J}\right)$ is a 4-vector as we have demonstrated, we can also define the 4-potential

$$A^\mu \equiv \left(\phi, \vec{A}\right)$$

Then, the wave equations are simply, in 4-form,

$$\partial^\alpha \partial_\alpha A^\beta = -4\pi J^\beta$$

And the Lorenz gauge condition

$$A^\alpha{}_{,\alpha} = 0$$

Reminder: $\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$

Gauge-free form

Recall that before we took the Lorenz gauge, our equations for the potentials looked like:

$$\nabla^2 \Phi + \frac{\partial(\nabla \cdot \boldsymbol{A})}{c\,\partial t} = -4\pi \rho_q$$

$$\nabla^2 \boldsymbol{A} - \frac{\partial^2 \boldsymbol{A}}{c^2\,\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{\partial \Phi}{c\,\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{J}$$

By applying the definitions $J^\mu = (\rho, \vec{J})$; $A^\mu \equiv (\phi, \vec{A})$; $\partial^\mu \partial_\mu = (\nabla^2 - \frac{\partial^2}{\partial t^2})$

We find the 4-form for the above 2 equations as $\partial^\alpha \partial_\alpha A^\beta - \partial^\beta \partial_\alpha A^\alpha = -4\pi J^\beta$

Since the equations for the potentials originally came from

$$\nabla \cdot \vec{D} = 4\pi\rho \qquad \nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

We should be able to manipulate $\partial^\alpha \partial_\alpha A^\beta - \partial^\beta \partial_\alpha A^\alpha = -4\pi J^\beta$ into something that gives us the Maxwell equations in terms of E and B fields

Towards the Maxwell equations

Lets rewrite $\partial^\alpha \partial_\alpha A^\beta - \partial^\beta \partial_\alpha A^\alpha = -4\pi J^\beta$ as $\partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -4\pi J^\beta$

Comparing with $\boxed{\nabla \cdot \vec{D} = 4\pi\rho}$ $\boxed{\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}}$

It should be clear that as the Maxwell equations contain only 1st derivatives, $\partial^\alpha A^\beta - \partial^\beta A^\alpha$ should be some 2nd rank tensor that contains that E and B fields.

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & \partial^0 A^1 - \partial^1 A^0 & \partial^0 A^2 - \partial^2 A^0 & \partial^0 A^3 - \partial^3 A^0 \\ -(\partial^0 A^1 - \partial^1 A^0) & 0 & \partial^1 A^2 - \partial^2 A^1 & \partial^1 A^3 - \partial^3 A^1 \\ -(\partial^0 A^2 - \partial^2 A^0) & -(\partial^1 A^2 - \partial^2 A^1) & 0 & \partial^2 A^3 - \partial^3 A^2 \\ -(\partial^0 A^3 - \partial^3 A^0) & -(\partial^1 A^3 - \partial^3 A^1) & -(\partial^2 A^3 - \partial^3 A^2) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} & -\frac{\partial A_y}{\partial t} - \frac{\partial \phi}{\partial y} & -\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z} \\ & 0 & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} & \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \\ & & 0 & \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$\boxed{\vec{B} = \nabla \times \vec{A}}$ $\boxed{\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}}$

The Maxwell's equations in 4-form

As we defined the Minkowski metric as $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Same in Griffiths, Introduction to Electrodynamics 3ed

the Faraday tensor has the form we derived

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \text{ and } F_{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

The source equation $\partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -4\pi J^\beta$ then becomes $\partial_\alpha F^{\alpha\beta} = -4\pi J^\beta$

Or, since $F^{\alpha\beta} = -F^{\beta\alpha}$ (2^{nd} rank anti-symmetric tensor), $\partial_\alpha F^{\beta\alpha} = 4\pi J^\beta \rightarrow \partial_\beta F^{\alpha\beta} = 4\pi J^\alpha$

The Maxwell's equations in 4-form (cont')

Also, we can define another 2nd rank tensor (sometimes called the Maxwell tensor)

$$G^{\alpha\beta} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Can also be expressed as $\partial_\beta G^{\alpha\beta} = 0$

Faraday tensor

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

Note that in Jackson, one finds

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Different by a minus sign

This is because Jackson uses the metric

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Looking at page 11 where we derived the Faraday tensor will show why.

The Maxwell's equations in 4-form (cont')

We can also show that $\partial_\beta G^{\alpha\beta} = 0$ can be written as $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

which basically is just an identity.

This means that by defining $F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$ we have already included the homogeneous Maxwell equations.

4-force

Previously, we have defined the equations $U^{\alpha} \equiv \frac{\mathrm{dx}^{\alpha}}{\mathrm{d}\tau}$; $P^{\alpha} \equiv m_0 U^{\alpha}$; $F^{\alpha} = \frac{\mathrm{dP}^{\alpha}}{\mathrm{d}\tau}$

Now, compare

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$\left[\frac{d\mathcal{E}}{dt}\right]_{\mathrm{EM}} = q\,\boldsymbol{V} \cdot \boldsymbol{E}$$

$$\boldsymbol{F}_{\mathrm{EM}} = q\left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \times \boldsymbol{B}\right)$$

Then one can see that what is left to write out the electromagnetic 4-force is U^{α}

Thus,

$$\frac{\mathrm{dP}^{\alpha}}{\mathrm{d}\tau} = \frac{q}{c} F^{\alpha\beta} U_{\beta}$$

Simultaneously contains the work done by the E field and the Lorenz force.

Equation 6.116 is wrong!! It should be the one-form 4-velocity

Plane EM waves in 4-form

Previously, we found that in the Lorenz gauge $\partial^\alpha \partial_\alpha A^\beta = -4\pi J^\beta$,

Therefore the source free equation is $\partial^\alpha \partial_\alpha A^\beta = 0$, which has the solution

$$\mathbf{A} = \Re\left\{\mathbb{A}\exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{x})\right]\right\}$$

We see that the phase term $k \cdot x = -\omega\mathrm{t} + k_x x + k_y y + k_z z$

Then, since phase is also a scalar, (sorry I don't have a very good explanation for this), we can rewrite $k \cdot x = -\omega\mathrm{t} + k_x x + k_y y + k_z z = \eta_{\alpha\beta} k^\alpha x^\beta$

$$k^\alpha = (\omega, \vec{k})$$

$\partial^\alpha \partial_\alpha A^\beta = 0$ then gives $\eta_{\alpha\beta} k^\alpha k^\beta = -\omega^2 + k^2 = 0$ $\rightarrow$ photons travel along null vectors

Fluid aspects next week